31^{st} International Mathematical Olympiad

Beijing, China

Day I

July 12, 1990

1. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If
$$\frac{AM}{AB} = t,$$
find
$$\frac{EG}{EF}$$
in terms of t.

2. Let $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

3. Determine all integers $n > 1$ such that
$$\frac{2^n + 1}{n^2}$$
is an integer.

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4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}$$
for all x, y in $\mathbb{Q}^+$.

5. Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, ... alternately according to the following rules:

   Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that
$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$
   Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that
$$\frac{n_{2k+1}}{n_{2k+2}}$$
   is a prime raised to a positive integer power.

   Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does:

   (a) $\mathcal{A}$ have a winning strategy?

   (b) $\mathcal{B}$ have a winning strategy?

   (c) Neither player have a winning strategy?

6. Prove that there exists a convex 1990-gon with the following two properties:

   (a) All angles are equal.

   (b) The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, ..., 1990^2 in some order.